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CITATION:

高橋, 眞映. The proximinality of the center of a quasicentral C^*-algebra (Harmonic/analytic function spaces and linear operators). 数理解析研究所講究録 1998, 1049: 94-102

ISSUE DATE:

1998-06

URL:

http://hdl.handle.net/2433/62200

数理解析研究所講究録
1049巻 1998年 94-102

The proximinality of the center of a quasicentral C*-algebra

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Abstract. Let A be a quasicentral C*-algebra and Z(A) its center. If the maximal ideal space of Z(A) is σ–compact and paracompact, then Z(A) is a proximinal subspace of A.

1. Banach空間Xの部分空間Yは、Xの各点からYへの距離を実現するYの点があるとき、proximinalであると言う。有限次元部分空間は常にproximinalであるが、一般の部分空間が常にproximinalであるとは限らない。今XをC*-環と言う具体的なBanach空間に限って話しを進める。Akermann - Perderson - Tomiyama [1]はC*-環の任意の閉両側イデアルはproximinalであることを示した。またOlech [8]は単位的可換C*-環の任意の同じ単位元を含むC*-部分環はproximinalであることを示した。C*-環の中心への距離は内部微分の作用素ノルムと関連して、大事な概念であるが、Somerset [11]はC*-環の中心のproxinimalityに興味を持ち、単位的なある種のC*-環の中心はproximinalであることを示した。最近彼は[12]で任意の単位的C*-環の中心はproximinalであることを示した。彼の手法はGlimmイデアルを用いて中心への距離をはかり、Stampfli [13]の作用素Pythagorean relationをうまく利用して、Michael [7]のcontinuous selection theoremに持ち込むと言うものであった。しかし非単位的な場合は、一般には彼の手法をそのまま使うことはできない。そこで彼の手法を利用出来そうな非単位的C*-環はないかと考えた。あるとき、Archbold [1]によって導入された擬中心的C*-環が、もしかしたらそのようなC*-環であろうと直観した。これはどんな原始イデアルも中心を含まないようなC*-環を言う (cf. [14])。我々はSomersetの手法を利用出来るように工夫し、先ず擬中心的C*-環の場合も、中心への距離が彼と同様な計算式で与えられることを示す。次に、その計算式及びCohen [3]のfactorization theoremを用い、彼の手法に沿って次の結果を示す。

定理。擬中心的C-環の中心は、その極大イデアル空間がσ–コンパクト且つパラコンパクトであるとき、proximinalである。*

これはC*-環が単位的であればSomersetの定理そのものであることに注意する。またこれらの問題はもっと一般のBanach moduleで考えるべきではないかと考えている (cf. [15])。

2. 以後特に断わらない限りAを擬中心的C*-環、Z(A)をその中心、$\Phi_{Z(A)}$をZ(A)の極大イデアル空間とする。各$\varphi\in\Phi_{Z(A)}$に対して、$\hat{e}_\varphi(\varphi)=1,\ 0\le\hat{e}_\varphi\le 1$なる$e_\varphi\in Z(A)$を一つ選んでおく。ここに^はGelfand変換を表す。$\tilde{A}=A+C\cdot 1$をAに単位元を添加して得られるC*-環とする。このとき、容易な観察から、その中心$Z(\tilde{A})$は$Z(A)+C\cdot 1$に等しいことが分かる。各$\varphi\in\Phi_{Z(A)}$に対して、

$\tilde{\varphi}(x+\lambda\cdot 1)=\varphi(x)+\lambda\ \ (x+\lambda\cdot 1\in\tilde{A})$ ，更に $\tilde{0}(x+\lambda\cdot 1)=\lambda\ \ (x+\lambda\cdot 1\in\tilde{A})$ とおいて、写像：$\varphi\rightarrow\tilde{\varphi}$ は $\Phi_{Z(A)}\cup\{0\}$ ($\subseteq A^*$) から $\Phi_{Z(\tilde{A})}$ への同相写像を与える。それ故 $\Phi_{Z(\tilde{A})}$ は $\Phi_{Z(A)}$ の 1 点コンパクト化と考えられる。次の定理は Cohen の Factori zation Theorem と呼ばれるもので、本論の主要な手段の一つである。

3. Theorem (Cohen [3]). Let B be a Banach algebra with a left approximate identity bounded by $K\geq 1$ and let X be a left Banach B-module. Then for every $z\in X_e$ and $\varepsilon>0$ there exist elements $a\in B$ and $y\in X$ such that $z=ay, \|a\|\leq K, y\in\overline{Bz}, \|y-z\|<\varepsilon$, where X_e is the closed linear subspace of X spanned by BX, which is called the essential part of X.

4. 各 $\varphi\in\Phi_{Z(A)}$, に対して、$G_\varphi=G_\varphi(A)$ を Glimm ideal, つまり、$Ker\ \varphi$ の生成する A の両側イデアルとする。今 A を left Banach $Ker\ \varphi$-module とみて、その essential part は Theorem 2 から $G_\varphi=\overline{G_\varphi}=(Ker\ \varphi)A$ となっている。

Lemma. $e_\varphi+G_\varphi$ is the identity element of A/G_φ .

Proof. Let $x\in A$. Since A is quasicentral, it folows that there exist $z\in Z(A)$ and $a\in A$ such that $x=za$ by tTheorem 2. Therefore $e_\varphi z-z\in Ker\ \varphi\subseteq G_\varphi$ and hence $e_\varphi x-x=(e_\varphi z-z)a\in G_\varphi$. This means that $e_\varphi+G_\varphi$ is the identity element of A/G_φ . Q. E. D.

5. 次は [9, Theorem 2.7. 5] の直接の結果である。但し Prim A は A の原始イデアル全体のつくる構造空間を表す。

Lemma . The mapping：

$$\begin{array}{ccc} \text{Prim } A & \rightarrow & \Phi_{Z(A)} \\ \Psi & & \Psi \\ P & \rightarrow & \varphi_P \end{array} \qquad (Ker\varphi_P=P\cap Ker\ \varphi)$$

is continuous and surjective.

In particular, $\bigcap_{\varphi\in\Phi_{Z(A)}} G_\varphi=\{0\}$ and henc、$\|x\|=\sup_{\varphi\in\Phi_{Z(A)}}\|x+G_\varphi\|$ for all $x\in A$.

6. $\hat{A}$ を A のゼロでない既約表現の同値類のクラスとする（Jacobson 位相から導かれる位相を入れたものを、A のスペクトラムと呼ぶ。）$\pi\in\hat{A}$ に対して、$\tilde{\pi}(x+\lambda\cdot 1)=\pi(x)+\lambda I_{H_\pi}$, $\omega(x+\lambda\cdot 1)=\lambda I_C$ とおいて、$(\tilde{A})^\wedge=\{\tilde{\pi}:\pi\in\hat{A}\}\cup\{\omega\}$ である。

7. Lemma. $\lambda e_\varphi+G_{\tilde{\varphi}}=\lambda\cdot 1+G_{\tilde{\varphi}}\ \ (\lambda\in C, \varphi\in\Phi_{Z(A)})$.

Proof. Let $\lambda\in C$ and $\varphi\in\Phi_{Z(A)}$. Then $\tilde{\varphi}(\lambda e_\varphi)=\varphi(\lambda e_\varphi)=\lambda=\tilde{\varphi}(\lambda\cdot 1)$ and hence $\lambda e_\varphi-\lambda\cdot 1\in Ker\ \tilde{\varphi}\subseteq G_{\tilde{\varphi}}$. Q. E. D.

8. Lemma. $\|x+G_\varphi\|=\|x+G_{\tilde{\varphi}}\|$ for all $x\in A$ and $\varphi\in\Phi_{Z(A)}$.

Proof. Let $x \in A$ and $\varphi \in \Phi_{Z(A)}$. Note that $G_\varphi \neq A$. In fact, if $G_\varphi = A$, then $Z(A) \subseteq G_\varphi$. By Lemma 5, there exists $P \in \mathrm{Prim}\, A$ such that $P \cap Z(A) = Ker\ \varphi$ and hence $Ker\ \varphi \subseteq P$, so $A = G_\varphi \subseteq P$, a contradiction. Therefore there exists an element $\rho \in (A / G_\varphi)^\wedge$ such that $\left|\rho(x + G_\varphi)\right| = \left|x + G_\varphi\right|$ (see [4, Lemma 3.3.6). But there exists a unique element $\pi \in \hat{A}$ such that $G_\varphi \subseteq Ker\ \pi$ and $\rho(a + G_\varphi) = \pi(a)$ for all $a \in A$. We assert that $G_{\tilde{\varphi}} \subseteq Ker\ \tilde{\pi}$. In fact, for each $z \in Z(A)$, we can find a uniue complex number $f(z)$ such that $\pi(z) = f(z) I_{H_\pi}$, since π is irreducible. Then f is a homomorphism. Also $f \neq 0$. If not, then $Z(A) \subseteq Ker\ \pi$ and this contradicts the quasicentrality of A. Thus $f \in \Phi_{Z(A)}$. Moreover, $Ker\ f \subseteq Ker\ \pi \cap Z(A) \in \mathrm{Prim}\ Z(A)$ and so $Ker\ f = Ker\ \pi \cap Z(A)$. But $Ker\ \varphi \subseteq G_\varphi \subseteq Ker\ \pi$ and so $Ker\ \varphi = Ker\ \pi \cap Z(A)$. Thus $Ker\ \varphi = Ker\ f$, so $\varphi = f$. Therefore if $z + \lambda \cdot 1 \in Ker\ \tilde{\varphi}$, then

$$\tilde{\pi}(z + \lambda \cdot 1) = \pi(z) + \lambda I_{H_\pi} = \varphi(z) I_{H_\pi} + \lambda I_{H_\pi} = \tilde{\varphi}(z + \lambda \cdot 1) I_{H_\pi} = 0,$$

so that $z + \lambda \cdot 1 \in Ker\ \tilde{\pi}$. Then $Ker\ \tilde{\varphi} \subseteq Ker\ \tilde{\pi}$ and hence $G_{\tilde{\varphi}} \subseteq Ker\ \tilde{\pi}$. It follows that

$$\left|x + G_\varphi\right| = \left|\pi(x)\right| = \left|\tilde{\pi}(x)\right| = \left|x + Ker\ \tilde{\pi}\right| \leq \left|x + G_{\tilde{\varphi}}\right|.$$

On the other hand, since $G_\varphi = (Ker\ \varphi)A \subseteq (Ker\ \tilde{\varphi})\tilde{A} = G_{\tilde{\varphi}}$, it follows that $\left|x + G_\varphi\right| \geq \left|x + G_{\tilde{\varphi}}\right|$. Q. E. D.

9. Lemma. Let $x \in A$ and $\lambda \in C$. Then the mapping : $\varphi \to \left|(x + \lambda e_\varphi) + G_\varphi\right|$ is upper semi-continuous on $\Phi_{Z(A)}$.

Proof. Let $x \in A$, $\lambda \in C$ and $\alpha > 0$, and set

$$G(\alpha, x, \lambda) = \{\varphi \in \Phi_{Z(A)} : \left|(x + \lambda e_\varphi) + G_\varphi\right| < \alpha\}.$$

$$\tilde{G}(\alpha, x, \lambda) = \{\psi \in \Phi_{Z(\tilde{A})} : \left|(x + \lambda \cdot 1) + G_\psi\right| < \alpha\}.$$

Then $\tilde{G}(\alpha, x, \lambda)$ is an open subset of $\Phi_{Z(\tilde{A})}$ by [11, Proposition 1.1]. Also if $\varphi \in \Phi_{Z(A)}$, then

$$\begin{aligned}\left|(x + \lambda \cdot 1) + G_{\tilde{\varphi}}\right| &= \left|(x + G_{\tilde{\varphi}}) + (\lambda \cdot 1 + G_{\tilde{\varphi}})\right| \\ &= \left|(x + G_{\tilde{\varphi}}) + (\lambda e_\varphi + G_{\tilde{\varphi}})\right| \text{ (by Lemma 7)} \\ &= \left|(x + \lambda e_\varphi) + G_{\tilde{\varphi}}\right| \\ &= \left|(x + \lambda e_\varphi) + G_\varphi\right| \text{ (by Lemma 8)}.\end{aligned}$$

This implies that $(G(\alpha, x, \lambda))^\sim = \tilde{G}(\alpha, x, \lambda) \setminus \{\tilde{0}\}$ and hence $G(\alpha, x, \lambda)$ is an open subset of $\Phi_{Z(A)}$. Thus the mapping : $\varphi \to \left|(x + \lambda e_\varphi) + G_\varphi\right|$ is upper semi-continuous on $\Phi_{Z(A)}$.

Q. E. D.

10. Lemma. If A is quasicentral, then $G_\varphi \cap Z(A) = Ker\ \varphi$ for each $\varphi \in \Phi_{Z(A)}$.

Proof. Assume that A is quasicentral and let $\varphi \in \Phi_{Z(A)}$. Choose $P \in \operatorname{Prim} A$ such that $G_\varphi \subseteq P$. Then $G_\varphi \cap Z(A) \subseteq P \cap Z(A) \neq Z(A)$ by the quasicentrality of A. But since $Ker\ \varphi \subseteq G_\varphi \cap Z(A)$, it follows that $Ker\ \varphi = G_\varphi \cap Z(A)$.

11. Lemma. Let $x \in A$ and $\alpha > 0$. Then $\{\varphi \in \Phi_{Z(A)} : \left| x + G_\varphi \right| \geq \alpha\}$ is compact.

Proof. Let $x \in A$ and $\alpha > 0$. Set $K = \{\varphi \in \Phi_{Z(A)} : \left| x + G_\varphi \right| \geq \alpha\}$. Let $\{F_\lambda : \lambda \in \Lambda\}$ be a decreasing net of relatively closed non-empty subsets of K. For each $\lambda \in \Lambda$, set

$$J_\lambda = \bigcap_{\varphi \in F_\lambda} G_\varphi .$$

Also for each $\lambda \in \Lambda$, take an element φ_λ of F_λ and then

$$\left| x + J_\lambda \right| \geq \left| x + G_{\varphi_\lambda} \right| \geq \alpha . \qquad (*)$$

Set $J = \overline{\bigcup_{\lambda \in \Lambda} J_\lambda}$. Since $\{J_\lambda : \lambda \in \Lambda\}$ is a increasing net of closed two-sided ideals of A, it follows that J is a closed two-sided ideal of A. Also (*) implies that $\left| x + J \right| \geq \alpha$. Since $\alpha > 0$, we have $x \notin J$ and hence A/J is non-zero C*-algbera. By [4, Lemma 3.3.6], we can find $\rho \in (A/J)^\wedge$ such that $\left| \rho(x + J) \right| = \left| x + J \right|$. But there exists a unique element $\pi \in \hat{A}$ such that $J \subseteq Ker\ \pi$ and $\rho(a + J) = \pi(a)$ for all $a \in A$. Hence

$$\left| x + Ker\ \pi \right| = \left| \pi(x) \right| = \left| \rho(x + J) \right| = \left| x + J \right| \geq \alpha .$$

Choose $\varphi \in \Phi_{Z(A)}$ such that $Ker\ \pi \cap Z(A) = Ker\ \varphi$. Since $G_\varphi \subseteq Ker\ \pi$, it follows that $\left| x + G_\varphi \right| \geq \alpha$ and so $\varphi \in K$. But since

$$\begin{aligned} Ker\ \varphi &= Ker\ \pi \cap Z(A) \\ &\supseteq J \cap Z(A) \\ &\supseteq J_\lambda \cap Z(A) \\ &= \bigcap_{\psi \in F_\lambda} G_\psi \cap Z(A) \\ &= \bigcap_{\psi \in F_\lambda} Ker\ \psi \ \text{(by Lemma 10)} \end{aligned}$$

for all $\lambda \in \Lambda$, it follows that $\varphi \in \overline{F_\lambda}$ for all $\lambda \in \Lambda$. Hence $\varphi \in \bigcap_{\lambda \in \Lambda} F_\lambda$ because each F_λ is relatively closed in K. We thus obtain that K is compact. Q. E. D.

12. Let X be a normed space and Y a subspace of X. For each $x \in X$, set

$$\pi_Y(x) = \{y \in Y : d(x, Y) \equiv \inf_{u \in Y} \left| x - u \right| = \left| x - y \right|\}.$$

We say that Y is proximinal if $\pi_Y(x) \neq \phi$ for all $x \in A$. We also say that Y is Chebychev if $\pi_Y(x)$ consists of a single point for each $x \in A$.

13. Let T be an operator of the Banach space B(H) consisting of all bounded linear operators on a Hilbet space H and set

$$W_0(T) = \{\lambda \in C : (T\xi_n, \xi_n) \to \lambda \text{ where } \left| \xi_n \right| = 1 \text{ and } \left| T\xi_n \right| \to \left| T \right|\}.$$

We call $W_0(T)$ the maximal numerical range of T.

Theorem (Stampfli [13]). The following three conditions are equivalent:

(1) $0 \in W_0(T)$.

(2) $\|T\|^2 + |\lambda|^2 \le \|T + \lambda I_H\|^2$ for all $\lambda \in C$.

(3) $\|T\| \le \|T + \lambda I_H\|$ for all $\lambda \in C$.

In particular, CI_H is a Chebyshev subspace of B(H) and

$$\|T - \pi_{CI_H}(T)\|^2 + \|\lambda I_H - \pi_{CI_H}(T)\|^2 \le \|T - \lambda I_H\|^2$$

for all $\lambda \in C$.

14. Theorem. Let A be a quasicentral C*-algebra. If $x \in A$, then

$$d(x, Z(A)) = \sup_{\varphi \in \Phi_{Z(A)}} \left\|(x + G_\varphi) - \pi_{C(e_\varphi + G_\varphi)}(x + G_\varphi)\right\|.$$

Proof. Let $x \in A$ and α the value of the right side above. Then

$$\begin{aligned} \|x - z\| &\ge \|x + G_\varphi - (z + G_\varphi\| \\ &= \|x + G_\varphi - \hat{z}(\varphi)(e_\varphi + G_\varphi)\| \\ &\ge \|x + G_\varphi - \pi_{C\cdot(e_\varphi + G_\varphi)}(x + G_\varphi)\| \end{aligned}$$

for all $z \in Z(A)$ *and* $\varphi \in \Phi_{Z(A)}$. Hence $d(x, Z(A)) \ge \alpha$. To show the converse inequality, let $\varepsilon > 0$ and set $K = \{\varphi \in \Phi_{Z(A)} : \|x + G_\varphi\| \ge \alpha + \varepsilon\}$. We consider two cases:

(i) $K = \phi$. Since $\|x + G_\varphi\| < \alpha + \varepsilon$ for all $\varphi \in \Phi_{Z(A)}$, it follows that

$$d(x, Z(A)) \le \|x\| = \sup_{\varphi \in \Phi_{Z(A)}} \|x + G_\varphi\| \le \alpha + \varepsilon$$

and hence $d(x, Z(A)) \le \alpha$ as $\varepsilon \downarrow 0$.

(ii) $K \ne \phi$. By Lemma 11, K is a non-empty compact subset of $\Phi_{Z(A)}$. Let φ be any element of K. Then there exists a unique scalar λ_φ such that $\pi_{C(e_\varphi + G_\varphi)}(x + G_\varphi) = \lambda_\varphi(e_\varphi + G_\varphi)$ since $C\,(e_\varphi + G_\varphi)$ is a Chebychev subspace of A / G_φ by Theorem 13. Set $z_\varphi = \lambda_\varphi e_\varphi$ and so $\|(x - z_\varphi) + G_\varphi\| \le \alpha$. Also put

$$W_\varphi = \{\psi \in \Phi_{Z(A)} : \|(x - z_\varphi) + G_\psi\| < \alpha + \varepsilon\}.$$

Then $\varphi \in W_\varphi$ and W_φ is a open subset of $\Phi_{Z(A)}$ by Lemma 9. Thus W_φ is an open neighbourhood of φ. Take a relative compact open neighbourhood U_φ of φ such that $U_\varphi \subseteq W_\varphi$. Since K is compact, there exist elements $\varphi_1, \dots, \varphi_n \in K$ such that $\bigcup_{i=1}^{n} U_{\varphi_i} \supseteq K$. Let $\{f_1, \dots, f_n, f_\infty\}$ be a partition of the identity for the convering $\{U_{\varphi_1}, \dots, U_{\varphi_n}, \Phi_{Z(A)} \setminus K\}$. Since each U_{φ_i} is relative compact, it follows that f_{φ_i} vanishes at infinity and hence there is an element $u_i \in Z(A)$ such that $f_i = \hat{u}_i$. Set

$$z = u_1 z_{\varphi_1} + \dots + u_n z_{\varphi_n}.$$

For any element ψ of $\Phi_{Z(A)}$ we have

$$\left| x + G_\psi - (z + G_\psi) \right| = \left| \sum_{i=1}^{n} f_i(\psi)(x - z_{\varphi_i}) + G_\psi + f_\infty(\psi) x + G_\psi \right|$$
$$\leq \sum_{i=1}^{n} f_i(\psi) \left| (x - z_{\varphi_i}) + G_\psi \right| + f_\infty(\psi) \left| x + G_\psi \right|.$$

If $\psi \in K$, then

$$\left| x + G_\psi - (z + G_\psi) \right| = \sum_{\psi \in U_{\varphi_i}} f_i(\psi) \left| (x - z_{\varphi_i}) + G_\psi \right| \leq \alpha + \varepsilon.$$

If also $\psi \notin K$, then

$$\left| x + G_\psi - (z + G_\psi) \right| \leq \sum_{\psi \in U_{\varphi_i}} f_i(\psi) \left| (x - z_{\varphi_i}) + G_\psi \right| + f_\infty(\psi) \left| x + G_\psi \right|$$
$$\leq \sum_{\psi \in U_{\varphi_i}} f_i(\psi)\,(\alpha + \varepsilon) + f_\infty(\psi)\,(\alpha + \varepsilon)$$
$$\leq \alpha + \varepsilon.$$

This implies that $\left| x - z \right| \leq \alpha + \varepsilon$ and hence $d(x, Z(A)) \leq \left| x - z \right| \leq \alpha$ as $\varepsilon \downarrow 0$. Q. E. D.

15. Theorem (Michael [7]). Let Ω be a paracompact T_1-space and X a Banach space. Then every lower semi-continuous carrier for Ω to the family of non-empty, closed convex subsets of X admits a continuous selection.

16. Theorem. Let A be a quasicentral C*-algebra and Z(A) its center. Suppose that Z(A) satisfies the following two conditions: (i) $\Phi_{Z(A)}$ is paracompact. (ii) there exists an element $v \in Z(A)$ such that $\hat{v}(\varphi) > 0$ for all $\varphi \in \Phi_{Z(A)}$. Then Z(A) is a proximinal subspace of A.

Proof. Suppose $\Phi_{Z(A)}$ is paracompact and there exists an element $v \in Z(A)$ such that $\hat{v}(\varphi) > 0$ for all $\varphi \in \Phi_{Z(A)}$. Let $x \in A$ and set $\alpha = d(x, Z(A))$. We can without loss of generality assume that $\alpha = 1$. By Theorem 3, we can find elements $u \in Z(A)$ and $a \in A$ such that $x = ua$ and $\left| u \right| \leq 1$. For each $\varphi \in \Phi_{Z(A)}$, there exists a unique scalar λ_φ such that $\pi_{C(e_\varphi + G_\varphi)}(x + G_\varphi) = \lambda_\varphi(e_\varphi + G_\varphi)$ since $C\,(e_\varphi + G_\varphi)$ is a Chebychev subspace of A / G_φ by Theorem 13. Then $\left| (x - \lambda_\varphi e_\varphi) + G_\varphi \right| \leq 1$ by Theorem 14. Also

$$\left| \lambda_\varphi \right| \leq \left| x + G_\varphi \right| = \left| (u + G_\varphi)(a + G_\varphi) \right| = \left| \hat{u}(\varphi)(e_\varphi a + G_\varphi) \right| \leq \left| \hat{u}(\varphi) \right| \left| a \right|. \tag{1}$$

The first inequality follows from Theorem 13. Set

$$C_\varphi = \{ \lambda \in C : \left| (x - \lambda e_\varphi) + G_\varphi \right| \leq 1 \text{ and } \left| \lambda \right| \leq \left| a \right| \left| \hat{u}(\varphi) \right| + \hat{v}(\varphi) \}.$$

Then each C_φ is a non-empty, closed, convex subset of C. We prove that the set-valued map : $\varphi \to C_\varphi$ is lower semi-continuous on $\Phi_{Z(A)}$. Let U be any open subset of C and set $\Phi = \{ \varphi \in \Phi_{Z(A)} : C_\varphi \cap U \neq \phi \}$. To show that Φ is an open subset of $\Phi_{Z(A)}$, let φ_0 be any element of Φ. Choose $\lambda_0 \in C_{\varphi_0} \cap U$ and take an open ball $U(\lambda_0; \varepsilon)$ of radius

$\varepsilon\,(0<\varepsilon<1)$ centered on λ_0 such that $U(\lambda_0;\,\varepsilon) \subseteq U$. Set

$$\Phi_0 = \{\varphi \in \Phi_{Z(A)} : \left|(x-\lambda_0 e_\varphi) + G_\varphi\right| < 1 + \tfrac{\varepsilon^3}{8} \text{ and } |\lambda_0| < \left|a\right| |\hat{u}(\varphi)| + (1+\tfrac{\varepsilon}{2})\hat{v}(\varphi)\}.$$

Then $\varphi_0 \in \Phi_0$ since $\varepsilon > 0$ and $\hat{v}(\varphi_0) > 0$, and Φ_0 is open by Lemma 9. Let $\psi \in \Phi_0$ and put $\beta = 1 - \left|(x-\lambda_\psi e_\psi) + G_\psi\right|$ and so $0 \le \beta \le 1$ by Theorem 14. Also, Theorem 13 implies that

$$\left|(x+G_\psi) - \lambda_\psi(e_\psi + G_\psi)\right|^2 + |\lambda_0 - \lambda_\psi|^2 \le \left|(x+G_\psi - \lambda_0(e_\psi + G_\psi)\right|^2 < \left(1+\tfrac{\varepsilon^3}{8}\right)^2,$$

and hence

$$|\lambda_0 - \lambda_\psi|^2 < \left(1+\tfrac{\varepsilon^3}{8}\right)^2 - (1-\beta)^2 = \tfrac{\varepsilon^3}{4} + \tfrac{\varepsilon^6}{64} + 2\beta - \beta^2 . \tag{2}$$

We consider two cases:

(i) $\beta < \frac{\varepsilon^2}{4}$. It follows form (2) that $|\lambda_0 - \lambda_\psi|^2 < \frac{\varepsilon^3}{4} + \frac{\varepsilon^6}{64} + \frac{\varepsilon^2}{2} < \frac{49}{64}\varepsilon^2$ and hence $\lambda_\psi \in U(\lambda_0;\,\varepsilon)$. Also we have $\left|(x-\lambda_\psi e_\psi) + G_\psi\right| \le 1$ by Theorem 14 and $|\lambda_\psi| \le |\hat{u}(\psi)|\left|a\right|$ by (1), so that $\lambda_\psi \in C_\psi$. Then $\psi \in \Phi$.

(ii) $\beta \ge \frac{\varepsilon^2}{4}$. Set $\mu = (1-\frac{\varepsilon}{2})\lambda_0 + \frac{\varepsilon}{2}\lambda_\psi$. It follows from (2) that

$$|\lambda_0 - \mu|^2 = \tfrac{\varepsilon^2}{4}|\lambda_0 - \lambda_\psi|^2 \le \tfrac{\varepsilon^2}{4}\left(\tfrac{\varepsilon^3}{4} + \tfrac{\varepsilon^6}{64} + 2\beta - \beta^2\right) < \tfrac{\varepsilon^2}{4}\left(\tfrac{1}{4} + \tfrac{1}{64} + 1\right) = \tfrac{81}{256}\varepsilon^2,$$

and hence $\mu \in U(\lambda_0;\,\varepsilon)$. Also we have

$$\begin{aligned}
\left|(x-\mu e_\psi) + G_\psi\right| &\le (1-\tfrac{\varepsilon}{2})\left|(x-\lambda_0 e_\psi) + G_\psi\right| + \tfrac{\varepsilon}{2}\left|(x-\lambda_\psi e_\psi) + G_\psi\right| \\
&\le (1-\tfrac{\varepsilon}{2})\,(1+\tfrac{\varepsilon^3}{8}) + \tfrac{\varepsilon}{2}(1-\beta) \quad (\text{since } \psi \in \Phi_0) \\
&< (1-\tfrac{\varepsilon}{2})\,(1+\tfrac{\varepsilon^3}{8}) + \tfrac{\varepsilon}{2}(1-\tfrac{\varepsilon^2}{4}) \\
&= 1 - \tfrac{\varepsilon^4}{16} < 1 .
\end{aligned}$$

Moreover,

$$\begin{aligned}
|\mu| &\le (1-\tfrac{\varepsilon}{2})\,|\lambda_0| + \tfrac{\varepsilon}{2}|\lambda_\psi| \\
&< (1-\tfrac{\varepsilon}{2})\left(\left|a\right||\hat{u}(\psi)| + (1+\tfrac{\varepsilon}{2})\hat{v}(\psi)\right) + \tfrac{\varepsilon}{2}\left|a\right||\hat{u}(\psi)| \quad (\text{since } \psi \in \Phi_0 \text{ and by (1)}) \\
&= \left|a\right||\hat{u}(\psi)| + (1-\tfrac{\varepsilon^2}{4})\,\hat{v}(\psi) \\
&< \left|a\right||\hat{u}(\psi)| + \hat{v}(\psi) .
\end{aligned}$$

Then $\mu \in C_\psi$ and hence $\psi \in \Phi$.

This shows that Φ is an open subset of $\Phi_{Z(A)}$ and hence the set-valued map : $\varphi \to C_\varphi$ is lower semi-continuous on $\Phi_{Z(A)}$. Since $\Phi_{Z(A)}$ is paracompact, it follows from Theorem 15 that we can find a continuous complex-valued function f on $\Phi_{Z(A)}$ such that $f(\varphi) \in C_\varphi$ for all $\varphi \in \Phi_{Z(A)}$. Since $|f(\varphi)| \le \left|a\right||\hat{u}(\varphi)| + \hat{v}(\varphi)$ for all $\varphi \in \Phi_{Z(A)}$, the function f vanishes at infinity and so $f = \hat{z}$ for some $z \in Z(A)$. Moreover,

$$\begin{aligned} \|x - z\| &= \sup_{\varphi \in \Phi_{Z(A)}} \|(x + G_\varphi) - (z + G_\varphi)\| \\ &= \sup_{\varphi \in \Phi_{Z(A)}} \|(x + G_\varphi) - (\hat{z}(\varphi)e_\varphi + G_\varphi)\| \\ &= \sup_{\varphi \in \Phi_{Z(A)}} \|(x - f(\varphi)e_\varphi) + G_\varphi\| \\ &\leq 1 \ \text{(since } f(\varphi) \in C_\varphi \text{ for all } \varphi \in \Phi_{Z(A)}). \end{aligned}$$

Therefore we have $\|x - z\| = d(x, Z(A))$. Q. E. D.

17. Remarks. (i) If $\Phi_{Z(A)}$ is connected, then $\Phi_{Z(A)}$: paracompact $\Leftrightarrow$ $\Phi_{Z(A)}$: σ–compact.

(ii) If $\Phi_{Z(A)}$ is σ–compact, then there exists an element $v \in Z(A)$ such that $\hat{v}(\varphi) > 0$ for all $\varphi \in \Phi_{Z(A)}$.

(iii) If Z(A) is separable, then $\Phi_{Z(A)}$ is paracompact and there exists an element $v \in Z(A)$ such that $\hat{v}(\varphi) > 0$ for all $\varphi \in \Phi_{Z(A)}$.

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